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Transient development

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Abstract
A physically rigorous first principles quantitative assessment is made of the transient development of manufacturing projects as tools are made and applied to create final products. Output quantities and production rates are compared for different development histories applying existing technologies. Technical progress is excluded from projects, but no limit is set on the technology available. The effects of the division of labour are examined and the conditions for maximising output determined. Predictions and empirical facts are compared, from which it is concluded abductively that transient solutions provide quantitative descriptions of the development histories of manufacturing projects and industries.

JEL codes D4, D20, D24, E1, E19, E22, E23, E24, O4, O11, O12, O47

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1. Introduction

From the earliest times, members of the species *homo sapiens* or perhaps more pertinently *homo faber*[1] have used tools to improve their ability to survive. The improvement of existing and the invention of new tools continues to this day. Limits to the process are not apparent; simple extrapolation into the future suggests the possibility of unlimited expansion.[2] Since the industrial revolution, increase in the range and scale of tools has facilitated development from simple workshops to major industrial complexes. The application of new and existing tool designs to create output is the origin of economic development and growth. Economic analysis may then be seen as man's attempts to understand and describe the mechanisms involved in this continuing process as human behaviour and social structures respond to the pressures created by the deployment of new and existing technologies.

Despite the effort expended in attempting to understand this process, the outcome can only be described as piecemeal. Posing the question, 'What do economists really know?', Blaug (1998) concludes that the formalism in use is the underlying problem and that:

> "Economics has increasingly become an intellectual game played for its own sake and not for its practical consequences. Economists have gradually converted the subject into a sort of social mathematics in which analytical rigor as understood in math departments is everything and empirical relevance (as understood in physics departments) is nothing. If a topic cannot

[1] Byrne (2004, p.31) states 'Tool use is an important aspect of being human that has assumed a central place in accounts of the evolutionary origins of human intelligence'.

[2] Resource depletion will ultimately force limits upon some technologies. However as human understanding is incomplete and new technologies unpredictable, future development may proceed in unexpected directions. In 'The Tragedy of the Commons', Hardin (1968) discusses implications of living on a finite planet.

> be tackled by formal modeling, it is simply consigned to the intellectual underworld" (Blaug, 1998, p.12–13).

In a paper first presented in 2006, before the economic upheavals which followed, Lavoie (2008) concluded that neither neoclassical nor heterodox analyses provide valid theories of production. Subsequently the many inadequacies of established economic thinking have been demonstrated: by its failure to predict; to establish control over; or even to explain the mechanisms allowing the extreme swings in global economic performance experienced in 2007 and the years following. The inappropriate hypotheses of the many schools of economic thought were laid bare.

Setterfield and Suresh (2014, p.812), in examining the dichotomy between micro and macro levels of economic analysis, conclude a necessity "for the prosecution of successful macroeconomic analysis by appeal to first principles". Bigo and Negru (2014, p. 329) state "Since the start of the global financial crisis, economists have increasingly acknowledged failures in their discipline" and (p.341) conclude "many economists across the board have tended to reaffirm their position... advocating the development of newer, better mathematical models that this time, allegedly, achieve greater realisticness (i.e. achieve a closer match to reality), promising a greater ability to successfully predict".

Clearly a mathematically-sound, physically-valid analysis from first principles is being called for; the subject of this paper.

1.1. Time in economic theory

Despite economic cycles being the norm from the beginnings of the industrial revolution, major areas of economic thought present equilibrium as an appropriate basis for analysis. Blaug (1998, p.23) comments "indeed real business cycle theory is, like new classical macroeconomics, a species of the genus of equilibrium explanations of the business cycle (which would yesteryear have been considered an oxymoron)." The formal treatment of time is eschewed.

In their articles, *The Production Function and the Theory of Capital*, both Robinson (1953) and Solow (1955) express their concerns about the use of time in economic analysis. Robinson points out that time is unidirectional in the real world, and that some mathematical descriptions fail to reflect the fact. Solow (1955, p.102) expresses his concerns, "But the real difficulty of the subject comes... from the intertwining of past, present and future."

Robinson (1980) continues to voice her critical assessment of the treatment of time in economic analysis. Later, no longer maintaining the concerns of his earlier insight, Solow (1994, p.47) states "Substitution along isoquants is routine stuff."

Setterfield (1995, p.23–24) concludes "it is not easy to introduce historical time into economic models in a manner that is at once meaningful and tractable. This is surely explained by the very nature of historical time, which must, by definition, defy deterministic, structural modelling"[3] and notes "the unwillingness of economic theorists to confront issues relating to historical time in the context of economic models".

[3] The present analysis is deterministic. Appendix A provides a different explanation of the intractability.

Boland (2005) in reviewing a three volume collection of articles, *Time in Economic Analysis*, by Zamagni and Agliardi (2004), concludes, "Namely, how can we build economic models where time matters because it is irreversible?" expressing his belief that this collection demonstrates the failure of economic analysis to offer any meaningful answer.

Clearly a physically valid analysis of economic development through time is required.

1.2. Physical dimensions

The International System of Units (SI) published by the Bureau International des Poids et Mesures (2006: updated 2014), is the English translation of the authoritative French text which defines physical units. It is a statement of the best scientific understanding of the nature of units of measurement. It states that physical quantities are expressed as products of a numerical value and a physical unit. Standard physical units can be one of a number of base units or units derived from them. *All* derived units are formed from the base units in combinations which have *negative, zero or positive small integer powers*. Dimensional analysis provides techniques by which to apply these requirements and thereby confirm the theoretical validity of the mathematical equations being applied to describe physical phenomena. It is a critical tool ensuring that symbolism conforms to reality but is little used in economic analysis. Barnett (2004) and Mayumi and Giampietro (2010) point out that the requirements of dimensional analysis are frequently neglected in economic publications.

Attributing theoretical significance to arbitrary equations containing physical units with fractional powers, violates the requirements of dimensional analysis. That the equations are good descriptions of the numerical data is irrelevant. That is an argument by analogy without first proving the analogy to be valid. A valid analysis using appropriate physical units is required.

1.3. Production and aggregate production functions

Economic analysts have introduced a bewildering range of mathematical expressions by which the production process might be described. Humphrey (1997) describes production functions before Cobb-Douglas and acknowledges that his description is incomplete as he has excluded others. S. K. Mishra (2010) presents a more recent overview of production functions. Zellner and Ryu (1998) seek to develop even more forms of *generalised production functions*; a process appropriate for curve fitting rather than developing theoretical understanding.

Georgescu-Roegen (1970) provides a rigorous but abstract mathematical description of the processes he perceived relevant to the quantitative description of the production process. He stresses the basic requirement that scientific symbolism should correspond to the real world. Amongst the wide range of factors he recognises as significant are time, utilisation factor and maintenance; parameters frequently ignored in conventional analysis. From an examination of the manner in which economic theorists formulate production functions, he (1970, p.2) infers that an "analytical imbroglio" exists in that "returns to scale must be constant in absolutely every production process".

Robinson (1953, p.82) states "To treat capital as a quantity of labour time expended in the past is congenial to the production-function point of view, for it corresponds to the essential nature of capital regarded as a factor of production". Solow (1955, p.101) dismisses her suggestion by asserting that "This has a faintly archaic flavour"; though a few sentences earlier he asserts that labour and capital are measured in "unambiguous physical units", not appreciating that, for this assertion to be true, he requires capital to be labour-time.[4] Their different positions led into the *Cambridge Controversies*.[5]

In perceiving capital to be the appropriate parameter of analysis, time *per se* is excluded from production theory. *Calculus: One and Several Variables*, states

> "If the path of an object is given in terms of a time parameter t and we eliminate the parameter to obtain an equation in x and y, it may be that we obtain a clearer view of the path, but we do so at considerable expense. The equation in x and y does not tell us where the particle is at any time t. The parametric equations do" (Salas, Hille and Etgen, 2007, p.499).

The elimination of time from production functions should make their overall shape more apparent. However the range of empirical data is insufficiently wide to make proper determination without further theoretical justification. Invalid hypotheses well established in conventional literature introduce distortions into how reality is interpreted.

Allen (2012) provides the widest range of empirical data and notes (p.6) that "Neutral technical change is detected especially between 1880 and 1965" and that "The regressions show that the rate of productivity growth increased with the capital-labor ratio". However technology-in-use is introduced into production functions in different ways; depending on how the technology is conjectured to affect the performance of the production equipment. Three particular hypotheses are Hicks-neutral, Solow-neutral and Harrod-neutral which are respectively

$$y = Af(k,l), \qquad y = f(Ak,l), \qquad y = f(k,Al), \tag{1}$$

where y is the output rate being evaluated, A, the technology-in-use factor,[6] k, the capital value and l, the labour applied.

1.3.1. The intercept of the aggregate production function

The value of the intercept of the aggregate production function is not to the fore of economic analysis. S. K. Mishra (2010, p.8) notes "It is surprising, however, that modern economists never formulate a production function in which labor alone can produce something." Von Thünen clearly understood the significance of the intercept. Jensen (2016), in examining von Thünen's contributions to the theory of production functions (von Thünen, J. H. *Der isolierte Staat in Beziehung auf Landwirtschaft und Nationalökonomie*, (Hamburg 1826)), identifies (2016, p.7), in English and the original German, the mathematical term (2016, p.7, equation(8)) which corresponds to von Thünen's "product of a man without capital (Arbeitsprodukt eines Mannes ohne Kapital)."

[4] Appendix A ends with the corollary that by representing production functions as $y = f(a,k,l)$, dimensional validity requires the units of capital to be labour-time.
[5] The Cambridge Controversies are discussed in general by Cohen and Harcourt (2003) and Lazzarini (2011) amongst others.
[6] The *factor* A is applied to different quantities in these general equations of production functions. Therefore, it must be a scalar quantity and as such can have no theoretical significance despite the appellation. This implies that the equations themselves are merely curve fitting devices.

That the value of the intercept is greater than zero is readily proved. At some point in human development, man had no tools. Man now has tools. Therefore the first tools were created without the use of tools. Labour is therefore able to create some output without tools.

The above-zero value presents empirically. All twelve figures shown by Allen (2012, pp.3–12) have intercepts above zero. His figure 7 (2012, p.8), plotting output rate against a low-value range of capital per worker, plots a relationship which he comments (2012, p.6) has continued for over two centuries. Badunenko and Zelenyuk (2004, p.469) show above zero values for the intercept in their figure 1.

Others presume the intercept to occur at the origin. Kumar and Russell (2002, p.534) base their tripartite decomposition on this presumption and introduce the origin as a point on the production frontier in figures 5, 6 and 7 (2002, pp.538–539). Makieła (2014) uses Cobb-Douglas and translog models which force the equations through the origin. The approaches taken by Koop, Osiewalski and Steel (1999) and (2000) produce equations passing through the origin but with very different slopes at the intercept: figure 1 (1999, p.478) shows the intercept being reached by lines with slopes of approximately one; figures 2 to 7 (1999, pp.481–492) with slopes of zero; figures 1 to 5 (2000, pp.294–295) with slopes of infinity.

It is shown that the intercept is determined by equation (14) and its significance is discussed in section 4.5, "The aggregate production function" and in section 4.7, "The Verdoorn coefficient and the intercept of the production function".

1.4. The present analysis

In his 1993 Nobel Prize lecture, *Economic Performance Through Time* (North, 1994) introduces a class of economic analysis as

> "A theory of economic dynamics comparable in precision to general equilibrium theory would be the ideal tool of analysis. In the absence of such a theory we can describe the characteristics of past economies, examine the performance of economies at various times, and engage in comparative static analysis; but missing is an analytical understanding of the way economies evolve through time.
>
> A theory of economic dynamics is also crucial for the field of economic development. There is no mystery why the field of development has failed to develop during the five decades since the end of the second World War. Neo-classical theory is simply an inappropriate tool to analyze and prescribe policies that will induce development.... The very methods employed by neo-classical economists have dictated the subject matter and militated against such a development.... In the analysis of economic performance through time it contained two erroneous assumptions: (i) that institutions do not matter and (ii) that time does not matter.
>
> This essay is about institutions and time. It does not provide a theory of economic dynamics comparable to general equilibrium theory. We do not have such a theory.[1] [The footnote states: In fact such a theory is unlikely. I refer the reader to Frank Hahn's prediction about the future of economic theory (Hahn, 1991).]" (North, 1994, p. 359).

The present analysis is a member[7] of the set North expected to remain unrealised and is presented with the intent of meeting the proposition:

> "Economic hypotheses can be judged by their logical coherence, their explanatory power, their generality, their fecundity, and, ultimately, their ability to predict. Economists, like all scientists, are concerned with predictability because it is the ultimate test of whether our theories are true and really capture the workings of the economic system independent of our wishes and intellectual preferences" (Blaug, 1998, p. 29).

The analysis is organised in the following manner. A physically valid mathematical representation of the production process is developed using both algebraic and differential equations. Solutions, limiting values and optimal production paths are determined. Output rates are shown to be aggregative. The solutions provide the theoretical rationale explaining Kaldor's stylised facts. Predictions of previously unknown relationships and their significance are revealed by the solutions. They are tested against empirical evidence which shows them to be consistent with that evidence and thus leads inexorably to a single conclusion.

2. Mathematical representation

Transient Development is a precise description of the scope of the present analysis. From first principles, it seeks a quantitative description of manufacturing projects' *development in and through time*.[8] *Development* is applied as an operant definition to describe projects which create tools by applying existing knowledge and techniques, to produce a final output. Projects do not experience technical progress.

This is quantitatively equivalent to the qualitative approach discussed by Robinson (1971, p.255) where she describes, "A book of blueprints representing a 'given state of technical knowledge'... Time, so to say, is at right angles to the blackboard on which the curve is drawn. At each point, an economy is conceived to be moving from the past into the future."

An Inquiry into the Nature and Causes of the Wealth of Nations begins

> "The greatest improvement in the productive powers of labour, and the greater part of the skill, dexterity, and judgement with which it is any where directed, or applied, seem to have been the effects of the division of labour" (Adam Smith, 1776).

From this insight, he develops a qualitative analysis, perceiving the division of labour as specialisation into separate manufacturing processes which may be further divided into subsidiary processes.

[7] To the best of the author's knowledge, no other published analyses are available which apply physically rigorous analytical techniques to quantify the production process.
[8] In and through time is derived from titles of papers by Robinson (1980), "Time in Economic Theory" Boland (2005), "Economics in time vs time in economics: Building models so that time matters", and North (1994), "Economic performance through time".

The present analysis begins at exactly the same point as Adam Smith but proceeds quite differently. The initial ideas he presents are fundamental truths and provide basic abstractions for quantitative assessment of economic projects. In current terminology, the *productive powers of labour* may be regarded as productivity; the *division of labour* remains unchanged; the *greater part of the skill, dexterity, and judgement with which it is any where directed*, *or applied* may be seen to be the strategic choices about how human effort is allocated.

One further abstraction is required for a quantitative assessment of production; labour is divided into two parts, the first makes and maintains the production tools, and the second uses them to make a final product. Tools are created before they can be used and those in use require maintaining. The basic parameters quantifying the production process are:

1. The numbers of people making, maintaining and using tools.
2. The periods of time over which tools are being made, maintained and used.
3. The number and effectiveness of the tools in use.
4. The quantity of output.

2.1. The defining relationships

The present analysis is made on a *per capita* basis. The variables used are: h – the number of people; q – total quantity of output; t – time. Suffixes are: $_d$ – identifying tool-making and maintenance; $_p$ – identifying production of the final output. The physical units are the *natural units* of the problem space; established by the definition of the unit of quantity and initial values.

Idealising assumptions, definitions and relationships are:

1. Quantitative relationships are linear; returns are to scale.

2. Human effort is defined as the product of the number of people working and the period of time worked. Two parameterisations are possible:
   a) retain the separation of h and t.
   b) introduce labour-time as a single variable, $H \equiv ht$ – this, however, creates significant difficulties in solving the resulting equations – the complications are examined in Appendix A, "Labour-time".

3. Productivity p, is defined as the rate of change of output with respect to the effort applied, $h_p t$;

$$p = \frac{\partial^2 q}{\partial h_p \partial t}. \tag{2}$$

4. The level of technical progress is defined by the constant, a, its value determines the effectiveness of the tools being made. a is the rate of change of the productivity of the tools being made with respect to the effort expended in making them, $h_d t$;

$$a = \frac{\partial^2 p}{\partial h_d \partial t}. \tag{3}$$

5. Maintaining and replacing worn equipment is a necessary part of manufacturing processes. Experience shows that new equipment requires less repair and replacement of worn parts than equipment which has been extensively used. Routine maintenance is generally carried out at predetermined intervals. Replacement of worn parts tends to be on a less precisely specified but at statistically determinable intervals. Attempting to follow such specific patterns might be appropriate for simulating the detailed performance of particular projects but is inappropriate in the search for general understanding. The regularities noted suggest that the maintenance requirement is best idealised as being in direct proportion to the machine-time expended.

   A maintenance constant, *m*, is defined to be the ratio of the effort required to maintain the tools to the machine-time used in operation.

6. The natural unit of output is the quantity produced by one person, working without tools, for one unit of time.

7. As the analysis is made on *per capita* basis, $h_d + h_p = h = 1$. In principle there is no implied restriction in the proportions of effort expended, except that the highest priority is given to maintaining the tools as they are used. Within this constraint all valid allocations of effort are possible but most of these will have no formal solution.

8. Initial values for projects are $t = 0$, $p_0 = 1$ and $q_0 = 0$. The suffix $_0$ indicates values at time zero. These values and the natural unit of output establish the natural system of units for the problem space.

In science and engineering, it is a *necessary condition* that, *when equations are used to represent physical reality, they are dimensionally correct*. Failure to satisfy this condition is considered *proof* that any such representation is *wrong*. The variables relevant to the present analysis are a, h, q, t and m; m acts only to modify the effort available for tool making. So from Buckingham's theorem, systems represented by four physical properties are quantifiable by single dimensionless groups. Dimensionless groups appropriate to the development process are examined in appendix B, "Dimensional analysis".

As an analysis from first principles, no form is imposed upon the nature of manufacturing projects. All descriptions are provided by solutions to the equations.

3. Solutions

No general analytical solution exists for partial differential equations (2) and (3) when reified to describe projects in which *ad hoc* decisions change the proportions of effort being applied. However, two idealised development patterns with formal solutions are applicable.

An algebraic solution is available for projects consisting of two phases. An initial tool-making phase, during which all the production tools are made, followed by a second phase during which the tools are used and maintained whilst creating the final output. This is a typical development pattern in which factories are constructed, commissioned and operated to produce final products.

A solution from calculus is available when the proportions of effort, allocated to production and to tool-making and maintenance, are fixed. This allows the partial differential equations to be expressed as ordinary differential equations which have formal solutions. This pattern is an idealised description of projects for which production continues while tools are made and brought into use. Early artisans are likely to have followed such a pattern of development: making their own tools whilst continuing to make the products providing their livelihood. The successful would have expanded, training apprentices, building bigger workshops and introducing improvements which ultimately allowed craft workshops to become factories. Setting h_p constant allows the productivity definition, equation (2), to be written as the ordinary differential equation

$$\dot{q} = p h_p \tag{4}$$

3.1. Boundaries

The definitions adopted imply boundaries.

The no-development boundary occurs when production continues without tool making: output rates remain unchanged; $h_p = 1$; $p = 1$. Integrating equation (4) over the time interval $[0, t]$ gives

$$q = h_p t = 1 \cdot t, \qquad \therefore \frac{q}{t} = h_p = 1. \tag{5}$$

The upper output rate boundary occurs when all the effort available for tool making and maintenance is fully committed to maintaining the tools already in use; no further effort is available to make more. While maintenance levels may be reduced through technological improvement, maintenance free tools are impossible. The physically achievable boundary lies within that defined by the no-maintenance boundary. By setting the division of labour constant; partial differential equation (3) becomes the ordinary differential equation

$$\dot{p} = a h_d \tag{6}$$

which integrated over the time interval $[0, t]$ gives

$$p = 1 + a h_d t.$$

Substituting p from equation (7) into equation (4) gives

$$\mathrm{d}_q = h_p(1 + \, a h_d t)\mathrm{d}t \qquad \dot{q} = h_p(1 + a t h_d) \tag{8}$$

which integrated over the time interval $[0, t]$ gives the total output and the overall mean output rate respectively as

$$q = h_p t + \frac{1}{2} a h_d h_p t^2, \qquad \therefore \frac{q}{t} = h_p\left(1 + \frac{1}{2} a h_d t\right). \tag{9}$$

3.2. Solving the differential equations

By letting the function $\eta(t)$ represent the total effort applied to tool making, as a function of time, the total effort expended in tool making and maintenance, at time t, is

$$h_d t = \eta(t) + m\int_0^t \eta(t)\mathrm{d}t \tag{10}$$

which when differentiated with respect to time and rearranged gives

$$\dot{\eta}(t) + m\eta(t) = h_d \tag{11}$$

which multiplied by the integrating factor e^{mt} has the solution

$$\eta(t) = \frac{1}{m}h_d(1 - e^{-mt}). \tag{12}$$

Analogous to equation (7), productivity is

$$p = 1 + a\eta(t). \tag{13}$$

Substituting productivity from equation (13) into equation (4) gives

$$\dot{q} = h_p[1 + a\eta(t)] = h_p\left[1 + \frac{a}{m}h_d(1 - e^{-mt})\right] \tag{14}$$

which integrated over the time interval $[0, t]$ gives

$$q = h_p\left\{t + \frac{a}{m}h_d\left[t - \frac{1}{m}(1 - e^{-mt})\right]\right\} \tag{15}$$

and the overall mean output rate is

$$\frac{q}{t} = h_p\left\{1 + \frac{a}{m}h_d\left[1 - \frac{1}{mt}(1 - e^{-mt})\right]\right\}. \tag{16}$$

3.2.1. Development front

By substituting $(1 - h_d)$ for h_p in equation (14), the output rate development front as a function of h_d and t is

$$\dot{q} = (1 - h_d)\left[1 + \frac{a}{m}h_d(1 - e^{-mt})\right] \tag{17}$$

with output rate limit,

$$\lim_{t\to\infty}\dot{q} = 1 + (\frac{a}{m} - 1)\, h_d - \frac{a}{m}h_d^2. \tag{18}$$

3.2.2. Optimal output rates

Equation (15) maps the total output as a function of time. The allocation of effort which maximises the output quantity produced for the application of minimum effort is determined by the following procedure. Substituting $1 - h_d$ for h_p in equation (15) and rewriting gives

$$q = (1 - h_d)\left\{t + \frac{a}{m}\left[t - \frac{(1 - e^{-mt})}{m}\right]h_d\right\}$$

for which the first derivative with respect to h_d is

$$\frac{\mathrm{d}q}{\mathrm{d}h_d} = \frac{a}{m}\left[t - \frac{(1 - e^{-mt})}{m}\right] - t - 2\frac{a}{m}\left[t - \frac{(1 - e^{-mt})}{m}\right]h_d \qquad (19)$$

and the second derivative is

$$\frac{\mathrm{d}^2 q}{\mathrm{d}h_d^2} = -2\frac{a}{m}\left[t - \frac{(1 - e^{-mt})}{m}\right].$$

The sign of the second derivative is determined by $t - (1 - e^{-mt})m^{-1} \geq 0$ for all $0 < m < 1$ and $t \geq 0$. Therefore $\frac{\mathrm{d}^2 q}{\mathrm{d}h_d^2} \leq 0$ and so maximum output rates occur at values of h_d determined by setting $\frac{\mathrm{d}q}{\mathrm{d}h_d} = 0$ in equation (19) which gives optimum values for h_d as

$$h_d = \frac{1}{2}\left\{1 - \frac{m}{a\left[1 - \frac{(1 - e^{-mt})}{mt}\right]}\right\} \qquad (20)$$

with limiting values for h_d and h_p:

$$\lim_{t\to\infty} h_d = \frac{1}{2}\left\{1 - \frac{m}{a}\right\} \qquad \lim_{t\to\infty, a\to\infty} h_d = \frac{1}{2} \qquad (21)$$

$$\lim_{t\to\infty} h_p = \frac{1}{2}\left\{1 + \frac{m}{a}\right\} \qquad \lim_{t\to\infty, a\to\infty} h_p = \frac{1}{2} \qquad (22)$$

3.3. Algebraic solutions

Equation (7) describes the productivity reached along the no maintenance boundary, which corresponds to the situation before tools are brought into use. By making tools over the time interval $[0, t_k]$, utilising all available manpower, $h_d = 1$, the effort expended is $h_d t_k$, productivity of the tools produced is $p_k = 1 + at_k$. When these tools are used to produce the final output, they require a maintenance effort of mt_k which leaves $(1 - mt_k)$ units of manpower to create the final output. Thus the *per capita* idealisation implies an upper limit to the useful quantity of tools created and the total output is

$$q = \begin{cases} 0, & 0 \le t \le t_k \\ (1 - mt_k)(1 + at_k)(t - t_k), & t \ge t_k \end{cases} \tag{23}$$

and the overall mean output rate is

$$\frac{q}{t} = \begin{cases} 0, & 0 \le t \le t_k \\ (1 - mt_k)(1 + at_k)\left(1 - \frac{t_k}{t}\right), & t \ge t_k \end{cases} \tag{24}$$

with the limiting overall mean output rate of

$$\lim_{t\to\infty} \frac{q}{t} = (1 - mt_k)(1 + at_k).$$

4. Discussion

Wide ranging relationships, based rigorously on physical principles, have been derived. Proven mathematically, they are necessarily true. As appropriate descriptions of the development process, their validity is determinable by comparison of prediction with empirical fact.

The assumption that the coefficient, a, remains constant, excludes technical progress from individual project histories. Actual values associated with projects, however, are established by the types and quantities of equipment installed. If mathematical relationships are reified by reference to specific empirical data, then the level of technical progress and associated limiting values are those established by the equipment in use. With no constraint on the value of a, the analysis is applicable at every level of technical progress.

Comparison of economic theory with empirical evidence, in academic publication, is generally presented as an assessment of how closely some hypothetical mathematical relationship or model fits empirical data; frequently alternate hypotheses are advanced. Coefficients for each are determined by curve fitting. The closest fit is then selected as an appropriate description of reality. While this process provides relationships suitable for interpolation within the range of the data, over-interpretation implying theoretical validity allowing extrapolation beyond that range, is likely to provide distorted views of reality.

Appendix B, “Dimensional analysis”, shows that several dimensionless groups are available to represent different views of reality. Each provides a theoretically valid description of an aspect of the production process. Conventional analysis interprets some relationships detected in the empirical data as being paradoxical. The existence of several valid dimensionless groups provides a ready explanation. All forms should be detectable within the empirical data.

In the following discussion, it will be seen that some hypotheses of conventional analysis are falsified. Falsification *per se* is not the purpose of this discussion. Comment, on this, is restricted to an essential minimum. However, in order to use some published empirical data, an assessment is necessary to separate actual facts from views of those facts which have been distorted by inappropriate hypotheses.

4.1. Aggregation

The total output relationships of equations (5), (15) and (23) are aggregative; algebraic totals are valid mathematically. All relationships developed for individual projects are, therefore, equally applicable to manufacturing industries. Aggregate descriptions of production are based on a theoretically valid summation process.

By the mean value theorem, a description, within the mathematical forms presented, will exist for aggregated empirical data. This conclusion will not be explicitly reiterated; reference will be made, without further comment, to projects or industries whichever is appropriate in context.

The many arguments against the use of aggregate production functions, based on heterogeneity (Felipe and Fisher, 2003; Felipe and McCombie, 2014; Felipe and McCombie, 2013), must therefore be seen in context. If the numeraire (generally money) presents an affine transformation from the theoretically valid labour-time measurements then the resulting equations will be representative of the underlying reality and therefore economically useful. This is sufficient to explain the widespread and successful use of aggregate production relationships in macroeconomic analysis.

4.2. Stylised facts

Economic analysis accepts that the complexities of reality make difficult the comparison and full understanding of theory and fact. Without analyses from first principles, there are few generally agreed interpretations by which to compare or guide further analysis. Economic models are tested against stylised facts that are accepted, in general if not in detail, as demonstrably true.

Beginning “As regards the process of economic change and development in capitalist societies”, Kaldor (1961, pp.178–179) suggested six stylised facts to which economic models should conform. Jones and Romer (2010, p.2) state that after nearly fifty years the first five of Kaldor’s stylised “facts have moved from research papers to textbooks” and that currently “researchers are grappling with Kaldor’s sixth fact”.

All the facts, except Kaldor’s statement of the capital:output ratio, are qualitative. The present analysis is quantitative. Simple direct comparison is inappropriate, but if the axioms of this analysis are empirically valid then its solutions should provide the theoretical explanation for Kaldor’s observations.

Fact 1 “The continued growth in the aggregate volume of production and in the productivity of labour at a steady trend rate; no recorded tendency for a falling rate of growth of productivity.” *And*

Fact 2 "A continued increase in the amount of capital per worker, whatever statistical measure of 'capital' is chosen in this connection."

Both the output rate equation (14) and the productivity equation (13) contain $\eta(t)$. The final two paragraphs of appendix A, "Labour-time", demonstrate that the general production function implies capital to be the labour-time expended in tool making, and that it is the monotonically increasing function $\eta(t)$, equation (12). Therefore, both of Kaldor's stylised facts are realised by the monotonically increasing capital relationship, $\eta(t)$.

Fact 4 "Steady capital-output ratios over long periods; at least there are no clear long-term trends, either rising or falling, if differences in the degree of utilization of capacity are allowed for. This implies, or reflects, the near-identity in the percentage rates of growth of production and of the capital stock – i.e. that for the economy as a whole, and over longer periods, income and capital tend to grow at the same rate."

This *fact* may be summarised as *the output:capital ratio is stable over time*.

4.2.1. The stability of the output:capital ratio

Since Kaldor (1961, p.178), various other studies have provided a range of differing conclusions about the stability of this ratio.

D'Adda and Scorcu (2003, pp.1180–1181) present values for seven industrialised economies for years in the range 1890 to 1990. They interpret the values for the USA as matching Kaldor's stylised fact 4, although the values they show graphically are about 0.3 before World War II and are generally above 0.4 afterwards. While the data for the other countries exhibits various excursions before, they all show falling values after the second world war, from which they conclude that, except for the USA, "output–capital ratios, show a tendency to reduce progressively over time" (2003, p.1189).

Madsen, V. Mishra and Smyth (2012, p.214) plot output:capital ratios for sixteen OECD countries from 1870 to 2004. The values they present for the USA follow a similar pattern to, but with numerical values apparently much greater than, those presented by D'Adda and Scorcu. However, whilst citing the D'Adda and Scorcu paper, Madsen, V. Mishra and Smyth do not comment on the numerical differences. They conclude that depending upon differing hypotheses of structural breaks, conclusions as to whether the data represents trend or mean reversion changes but that neither determination is totally sustainable. However they lean towards the interpretation of the ratio being constant. This may be for the pragmatic reason that this "ease[s] the interpretation of factors that determine the balanced growth path" (Madsen, V. Mishra and Smyth, 2012, p.233).

Allen (2012, p. 6) examines production data between 1820 to 1990 and considers whether the capital:output ratio is constant and concludes that it is not. He determines a value of 0.59 for one constant which, if the ratio was actually constant, would have the value 1.0. The value determined, 0.59, is fifteen standard deviations below that of the 1.0 required to indicate the ratio to be constant.

Quite clearly, the empirical data, in both confirming stability and movement, present a reality which is more complex than simply testing whether the stability of the ratio provides a binary result. Transient analysis provides quantitative descriptions of the output rate, equation (14), and of capital, equation (12). The precise evaluation of the output:capital ratio is determined as follows.

Output:capital ratios for projects, expressed as the fraction $\frac{y}{k}$, are able to take on any value in the interval $[0, \infty]$. When output is being produced without the benefit of tools, capital is zero; the ratio is infinity. Initially when tools are made before being brought into use, no output has been produced; the ratio is then zero. In all economies, projects will be present at every stage of development. Stability can only be explained through aggregate values approaching the limiting values of the relevant relationships.

Appendix A concludes with the corollary that the concept of capital in production functions is the labour-time expended in producing the tools used. The output:capital ratio, $\frac{y}{k}$, is therefore

$$\frac{y}{k} \equiv \frac{\dot{q}}{\eta(t)} = \frac{h_p(1 + a\eta(t))}{\eta(t)} = h_p\left(\frac{1}{\eta(t)} + a\right) = h_p\left[\frac{m}{h_d(1 - e^{-mt})} + a\right] \tag{25}$$

$$\therefore \lim_{t\to\infty} \frac{y}{k} = h_p\left(\frac{m}{h_d} + a\right) \tag{26}$$

The limiting values represented by equation (26) remain in the interval $[0, \infty]$. However, as projects respond to competitive pressure the limiting values for h_d and h_p of equations (21) and (22), will be approached. Substituting these values into equation (26) gives

$$\lim_{t\to\infty} \frac{y}{k} = \frac{1}{2}\left(1 + \frac{m}{a}\right)\left[\frac{m}{\frac{1}{2}(1 - \frac{m}{a})} + a\right] = \frac{1}{2}\left(1 + \frac{m}{a}\right)\frac{a}{a}\left[\frac{2m + a - m}{\left(1 - \frac{m}{a}\right)}\right] = \frac{1}{2}\frac{(a + m)^2}{(a - m)} \tag{27}$$

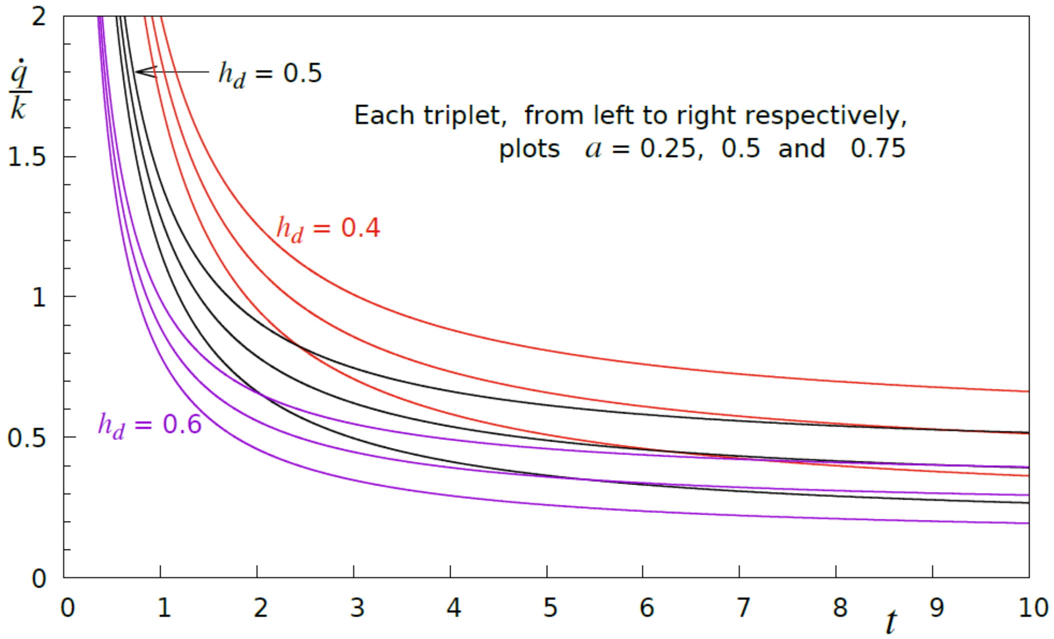
With continuing technical development, a will increase, so

$$\lim_{t\to\infty, a\gg m} \frac{y}{k} = \frac{a}{2} \tag{28}$$

The limiting values for the ratio will therefore demonstrate apparent stability over shorter time periods[9] and with increasing technical competence, a ratio increasing over generationally long time periods.

[9] Equation 31 shows the value of technical progress obtained from Table 1 to be 0.49. The corresponding limiting value for the output:capital ratio for US manufacturing industries from the Solow (1957) data is therefore 0.25. For the whole US economy, D'Adda and Scorcu (2003, p.1180), in their Figure 1, show comparable values of between 0.2 and just over 0.3.

Figure 1 Output:capital ratio as a function of time – equation (25)



However, empirical evidence is restricted to shorter periods of time, collected over many countries and industries, and with the effects of many independent decisions, both good and bad. The complexities of the output:capital ratio progression may be seen in Figure 1, which plots equation (25) for some possible combinations of the fraction of effort allocated to tool making and maintenance, h_d =0.4, 0.5 and 0.6 and for technology levels of a = 0.25, 0.5 and 0.75 natural units. Comparing the patterns shown in Figure 1 with those presented by Madsen, V. Mishra and Smyth' sub-figures of their Figure 1 (2012, p.214), shows that they all present similarities in their complexity, their decay with time and their convergence.

Equation (25) and the limiting value equations (27) and (28) provide a quantitative explanation of how the apparently inconsistent conclusions of the papers cited above arise: Kaldor (1961) (the ratio is effectively unchanging – true with no technical progress – consistent with equation (27)); D'Adda and Scorcu (2003, pp.1180–1181) (slowly increasing values for the USA – consistent with equation (28), the reducing values for the other countries – consistent with equation (25)); Madsen, V. Mishra and Smyth (mean reversion (stable ratio) – consistent with equation (27), trend reversion (increasing ratio) – consistent with equation (28)) and Allen (2012, p.6) (non-constant output-capital ratio – consistent with equation (28)).

Fact 6 expresses differences in the rate of growth of labour productivity and of total output in different societies whilst the other facts are maintained.

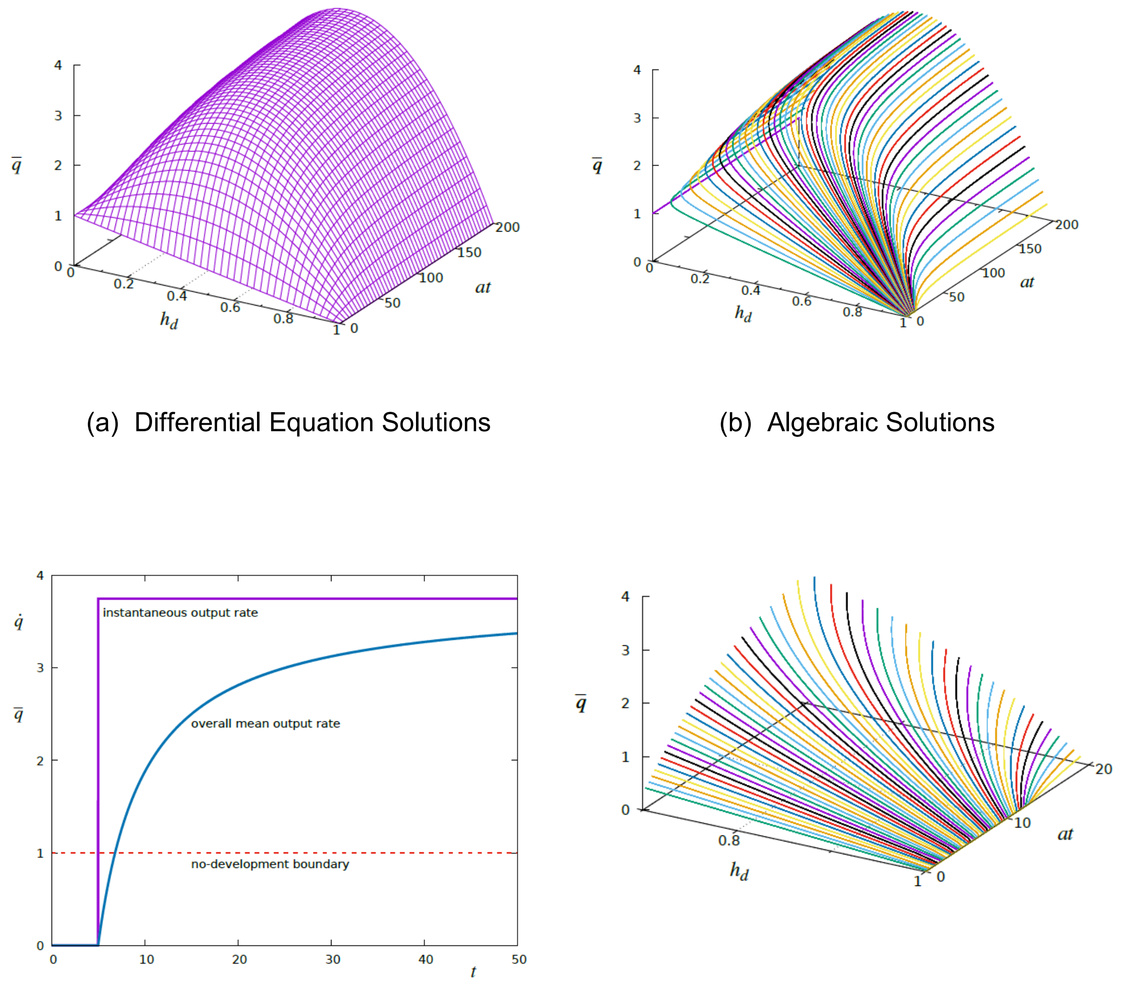
Conventional economic analysis, postulating equilibria, suggests unconditional convergence. The present analysis demonstrates that the performance of manufacturing industries depends upon decisions made about the quantity and the quality of the tools being created and applied. Inappropriate choices will produce less competitive industries; results consistent with Fact 6. A discussion of more recent research is considered in section 4.4, "Competition in manufacturing".

The third and fifth facts express relationships involving monetary values and as such they are directly outside the scope of the present analysis. However, with consistently distributed shares, both facts are in accordance with the steady capital-output ratio of Fact 4.

4.3. The development hypersurface

Equations (16) and (24) are the loci of all possible overall mean output rate histories of manufacturing projects and industries; the *development possibilities hypersurface*. Each strategy creates its own *world-line* traversing its own distinctive path over the hypersurface. Project-lines are the output rate histories of individual projects. Industry world-lines describe the movement of an industry's mean aggregated value over the hypersurface.

Figure 2 Overall mean output rate ($\bar{q}$) hypersurface views



(a) Differential Equation Solutions

(b) Algebraic Solutions

(c) Output Rates–Algebraic Solution

(d) Early Development–Algebraic Solutions

Projections of the development possibility hypersurface as functions of h_d and at[10] are shown in Figure 2. The values of the technical progress constant and the maintenance requirement are 1.0 and 0.075 respectively for each figure.

Project-lines for equation (16) are shown in Figure 2(a). The lines begin on the straight line joining points $(0,0,1)$ and $(0,0,0)$ and are at intervals of $\Delta h_d = 0.05$. Lines of constant at serve only to delineate the hypersurface.

The project-line trajectories of equation (24) are shown in Figures 2(b) and 2(d). Figure 2(d) enlarges a portion of Figure 2(b) to clarify the early development of the trajectories. A greater investment in tools, with a consequently later start of production, brings about very different project-line trajectories.

Equations (16) and (24) solve the same partial differential equations with very different development patterns, so while individual *project-line's* trajectories differ, they traverse the same hypersurface. This is clearly visible in Figures 2(a) and 1(b).

Figure 2(c) plots instantaneous and overall mean output rates along project-lines with production starting at $t_k = 5$. The dotted line is the output rate at the no-development boundary. For the initial tool investment case, at time t_k, the instantaneous output rate rises from zero and remains constant until production ends, whereas the overall mean output rate remains at zero until production starts and then converges asymptotically towards the value of the instantaneous output rate. Parameterisation into capital and labour implicitly introduces the instantaneous output rate view of reality and obscures the transient nature of the development process.

Contour plots, corresponding to Figures 2, are shown in Figure 3(a) for the development front equation (17), as functions of h_d and at, and for the limit equation (18), as a function of h_d. The development of the no-maintenance boundary is shown in 3(b).

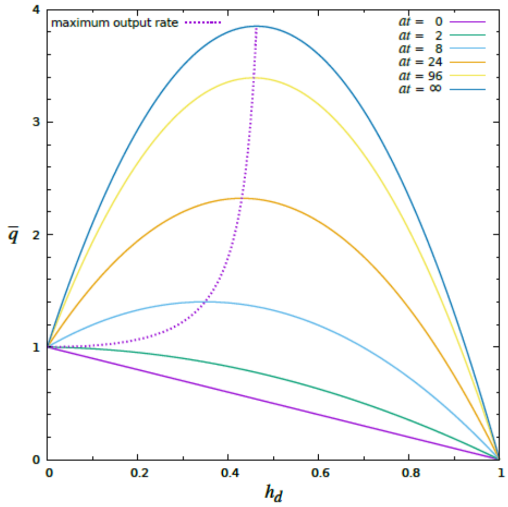
The no-development boundary, where no tools are being made, is the point $(0,1)$. At the point $(1,0)$, tools are being made but not used and so it represents the overall mean output rate of the algebraic equations before production begins, $t \leq t_k$.

The contours are points of equal effort expended. Assuming the efforts of tool makers and users have the same unit cost, it follows that these contours are the *isocost* lines of conventional analysis. Horizontal lines between the outermost points record the same overall mean output rate. So by definition these lines are *isoquants*.[11]

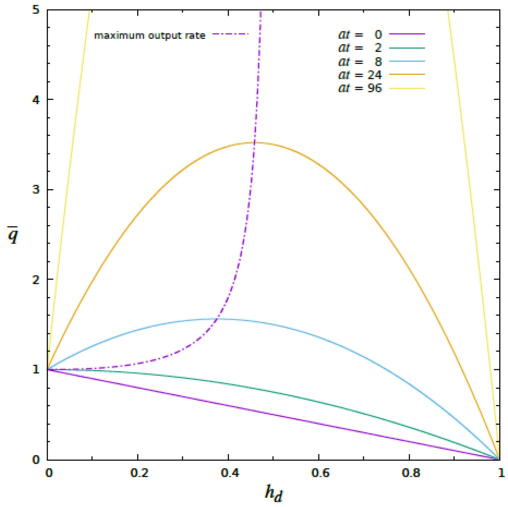
The monotonically increasing dashed curves, originating at point (0, 1), are the locus of points of maximum output rate with minimum effort. That of Figure 3(a) plots equation (20). It is the competitive growth path which competitive industries would have followed and it may be conjectured that this is the basis for the conventionally hypothesised *balanced growth path*.

[10] Equation (9) is the no-maintenance boundary but it is also the straight line from which equation (16) diverges under the effects of the maintenance decay term – at is therefore an appropriate parameter against which to map output rates.
[11] Isoquants have no meaningful existence in transient analysis. To provide a *realistic* description requires the introduction of concepts found in science fiction. Movement along an isoquant would be akin to travel in a time-machine for which movement through time requires simultaneous movement through alternate realities.

Figure 3 Overall mean output rate ($\bar{q}$) as a function of h_d and at



(a) With maintenance



(b) The no-maintenance boundary

4.4. Competition in manufacturing

In examining British productivity performance from the late-nineteenth to the early 21st century, Crafts (2012, p.18) notes "Economic theory gives somewhat ambiguous messages about the impact of competition on productivity performance" and that competition and good productivity performance moved together throughout the period. He (p.27) concludes "Applied economists in the UK are now generally agreed that strengthening competition in product markets is good for productivity performance."

Similar tendencies are reported on a worldwide basis:

> "I show in this article that unconditional convergence does exist, but it occurs in the modern parts of the economy rather than the economy as a whole. In particular, I document a highly robust tendency toward convergence in labor productivity in manufacturing activities, regardless of geography, policies, or other country-level influences" (Rodrik, 2013, p. 166).

Figure 3(a) provides the explanation for these observations. The most competitive industries would have followed the maximum output rate curve of equation (20). New techniques, embodying the latest technological advances, implying increasing values of a, would have been introduced as new projects were completed and brought into production. The mean value of a for the industry would have continued to increase, and with a the limiting values of equations (21) and (22) are changed.

The introduction of more productive equipment would have allowed greater output to be achieved more quickly. By a process of Darwinian selection those following this path most closely survived. Others fell by the wayside. At any level of technical development with appropriate apportioning of tool making and productive efforts, a single point of maximum output will be achieved. Through competition, unconditional convergence occurs.

At all levels of technical progress, the best engineering produces maximum productivity. Vergeer and Kleinknecht (2007, p. 20) confirm that low wages do not equate to competitiveness, by their conclusion, “Our panel data analysis shows that a causal link indeed exists between wage growth and labour productivity growth”.

Bénétrix, O’Rourke and Williamson (2015) examine the spread of manufacturing to the poor periphery between 1870 to 2007. They note that the industrial catching up on which they are reporting is different from the manufacturing productivity convergence reported by Rodrik (2013). They describe the catching-up process to be present throughout the period and that it reached a high point between 1950 and 1973. Again this is completely consistent with the predictions of the present analysis. There is no reason that the most appropriate engineering decisions will be made and thereby achieve competitiveness. Even valid choices may be undermined by other factors; inadequate infrastructure, civil unrest, corruption etc. There is no preferred direction of development; only better or worse choices.

Adam Smith’s metaphor of the *invisible hand* may be seen as an anthropomorphic interpretation of the pressure exerted by competitors’ increasing productivity. Good engineering increases competitive advantage. Competition enforces good engineering and allows output to be increased with no overall increase of effort. Economic competence and good engineering are the same. The invisible hand is the pressure created by others as they successfully introduce new and improved techniques into the production process.

The algebraic solution provides a description of a single manufacturing project which is built, brought into service, continues in production for a number of years and finally closes. This pattern may be aggregated to represent multiple projects, on single or many production sites, starting and finishing at different times and introducing the latest techniques; a scenario very familiar in advanced economies.

The solution through calculus provides a different scenario describing the earliest development patterns from household production to workshops operated by single artisans making their own tools to produce their wares. With the immediate feedback present at this scale the introduction of new effective techniques occurs organically by the selection of the best and discarding the less good. Again aggregation might be used to describe this process but it is unnecessary as calculus provides a direct description of the introduction of improving techniques, equation (20) shown in Figure 3(a). The *per capita* representation encompasses the increasing scale of production from workshop to factory. By the mean value theorem this solution is also the description of the development of competitive industries over extended periods of time. As the scale of industry increases even large plants are effectively infinitesimal in the expanding world of manufacturing.

4.5. The aggregate production function

Equation (14) is the transient relationship expressing the instantaneous output rate which is also evaluated by the production functions of conventional economic analysis. Through aggregation, it is comparable to the aggregate production function.

Solow (1957) analyses United States GNP data of private non-farm economic activity from 1909 to 1949 to obtain an assessment of the aggregate production function. He separates the form of the function from the effects of technical change. In commenting on this paper, McCombie (2000, p.271) observes that Solow "adopted a novel estimation procedure that has not been followed since".

Solow (1957, p.312) warns, "It will be seen that I am using the phrase 'technical change' as a shorthand expression for *any kind of shift* in the production function" and (p.320, footnote 18) "I have left Kendrick's GNP data in 1939 prices and Goldsmith's capital stock figures in 1929 prices". He explicitly acknowledges (p.314) that in using GNP, "the share of capital has to include depreciation". Solow's Table 1 contains numerical errors which were corrected by Hogan (1958, p.407). The corrected data are used for testing the present analysis.

Solow hypothesises that the general statement of the aggregate production function is $Q = A(t)\, f(K, L)$ and that by evaluating $A^{-1}(t)\Delta Q \Delta^{-1} t$, the effects of technical change may be separated so as to isolate the effects of the other factors. By applying this procedure he creates a numerical mapping of the function $\dot{q} = f(k, 1)$. In order to provide a mathematical description of the mapping, he (1957, p.318) observes, "As for fitting a curve to the scatter...I can't help feeling that little or nothing hangs on the choice of functional form, but I have experimented with several. In general I limited myself to two-parameter families of curves, linear in the parameters (for computational convenience), and at least capable of exhibiting diminishing returns (except for the straight line, which on this account proved inferior to all others)." The curves he fitted are plotted in Figure 4, over the capital range of zero to four, together with the matching transient relationship. The perimeter of Solow's chart 4, is indicated so that the range of the data may be seen in relation to the functional domain necessary for theoretical validity.

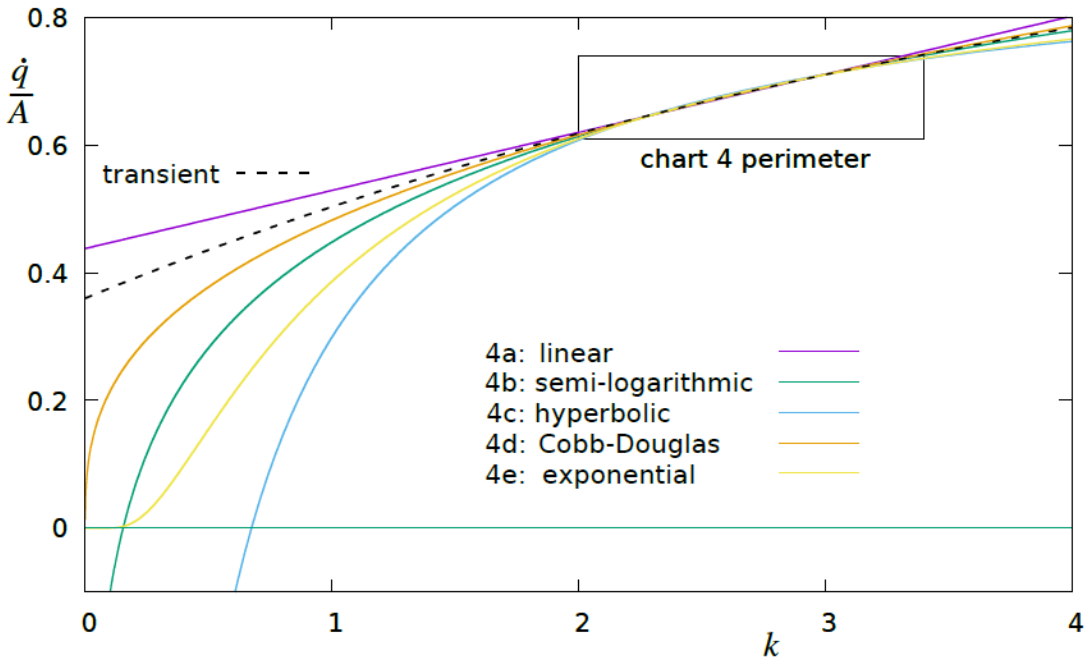
Despite starting (1957, p.319) "Thus any conclusions extending beyond the range actually observed in Chart 4 are necessarily treacherous", he expresses the full extent of the functional mapping verbally, describing the curves shown explicitly in Figure 4 with:

> "The particular possibilities tried were the following... Of these, (4d) is the Cobb-Douglas case; (4c and e) have upper asymptotes; the semi-logarithmic (4b) and the hyperbolic (4c) must cross the horizontal axis at a positive value of k and continue ever more steeply but irrelevantly downward (which means only that some positive k must be achieved before any output is forthcoming, but this is far outside the range of observation); (4e) begins at the origin with a phase of increasing returns and ends with a phase of diminishing returns—the point of inflection occurs at $k = \beta/2$ and needless to say all our observed points come well to the right of this" (Solow, 1957, p. 318).

The most notable omission from his list of fitted equations is equation (4a). It is the *linear relationship*, the only one having a *positive intercept* with $k = 0$ and therefore the *only possible contender* which *might have theoretical validity*.

The most notable assertion in this list is "*means only that some positive k must be achieved before any output is forthcoming*", appealing to *transient concepts in order to justify behaviour impossible in his universe of discourse!*

Figure 4 Solow's Chart 4 with the fitted curves and the transient approximation

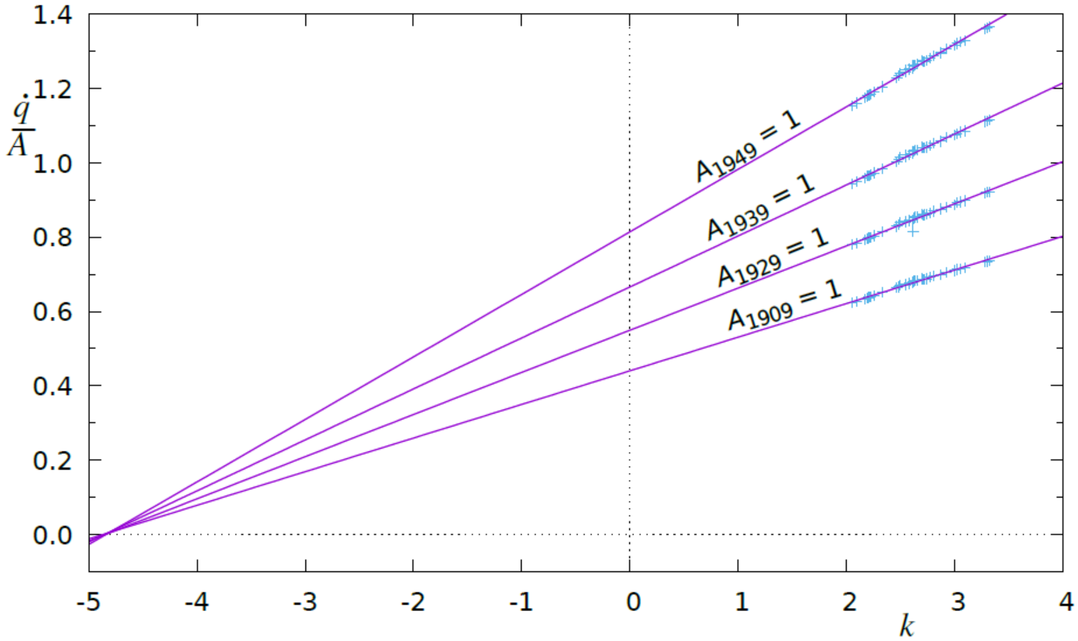


In responding to Hogan (1958), Solow (1958, p.412) notes "I can always choose q-units so that $A\,(t)\ =\ 1$ in some specified initial year" but he does not expand further nor discuss the implication that this assertion reduces his presentation of *the aggregate production function* to be merely a single possibility, arbitrarily selected, from the *forty-one alternatives* available!

The implications of this multiplicity are to be seen in Figure 5. $A\,(t)$ is set equal to one and applied to each of four years across the full range of the data. The years selected are: 1909, the first year of the data and the year Solow presents as *the aggregate production function*; 1929, the capital stock price basis of the data and the year for which he notes an increase in the rate of change of technical progress; 1939, the price basis for the GNP data and the start of World War II; 1949, the final year of the data. The mapped data points and linear fits to them are plotted over an extended *per capita* capital range, $-5 \leq k \leq 4$. The extension beyond the point of convergence allows recognition that the transformations are essentially affine. The mappings demonstrate that these manipulations create *four very different views* of the aggregate production function.

In replying to Hogan, Solow (1958, p.412) discusses the possibility that "a production function net of technical change... *wiggles and all*... will pass through the right points with the right outputs and with the right slope" but he does not appear to have carried out such an exercise. However, all is not lost. Transient analysis provides a method by which the mean aggregate production function for the data may be ascertained.

Figure 5 Solow's data – linear fits – $A(t) = 1$ for the years shown



The output rate relationship used by Solow is $\dot{q} = A\,(t)\,f\,(k, l)$. Equation (29) expresses this relationship without technical change and equation (30) is the equivalent transient relationship.

$$\dot{q} = f(k, l) \qquad \dot{q} = f(0,1); k = 0 \tag{29}$$

$$\dot{q} = (1 + a\eta(t))h_p \qquad \dot{q} = h_p; \quad t = 0 \tag{30}$$

The value of the intercept with $k = 0$ is determined by the simultaneous solution of these relationships: the mappings of the transformed numerical data representing equation (29); and the particular forms of equation (30) which correspond to the mappings.

For each base year mapping, an initial estimate of h_p is obtained from the intersection of the linear fit with $k = 0$ and of a from the slope of the fitted line.[12] An initial value of m is set to some likely fraction of effort; say $m = 0.1$. These three parameters effect the position, slope and curvature of the transient function being mapped: h_p is the point of intersection with the y axes; a determines the slope at the intercept of the transient relationship and of the no-maintenance boundary; m determines the curvature and hence how the curve separates from the no-maintenance boundary. The parameters are adjusted iteratively so that the resulting curve matches the transformed data points.

Figure 6 shows the transient relationships, the no-maintenance boundary and the data points transformed from capital into time by implementing the above procedure for the four years plotted in Figure 5.

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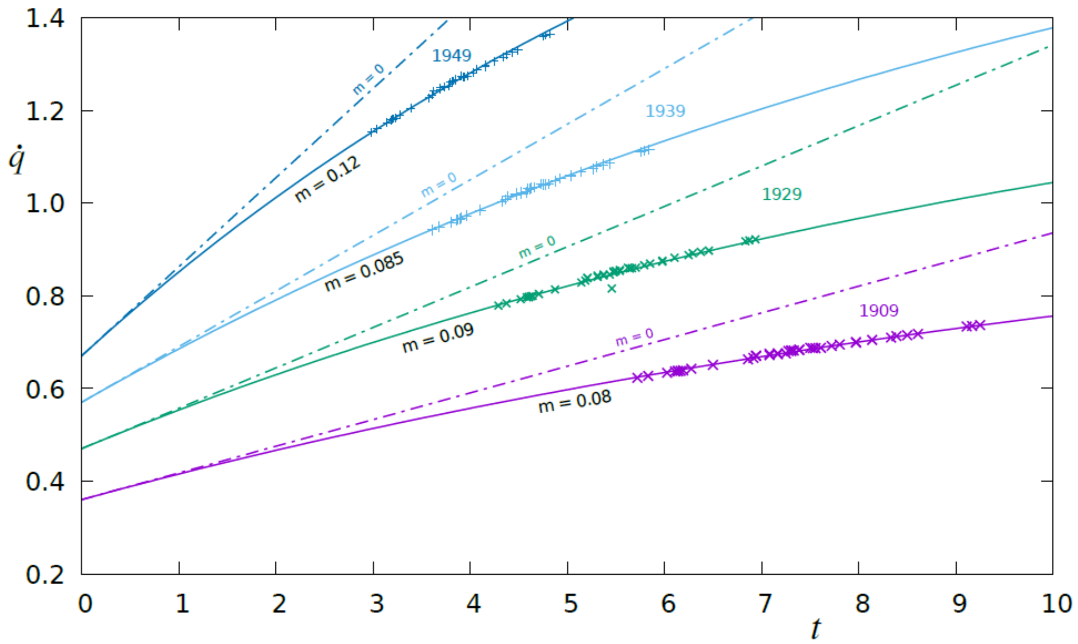
[12] Differentiating equation (8) with respect to time gives $\ddot{q} = ah_d h_p$ which rearranged is $a = \ddot{q}[1 - h_p)h_p]^{-1}$.

Figure 6 Solow data – instantaneous output rates as functions of time



An estimate of the mean transient function describing the Solow data may now be obtained. Historical events determined the actual proportions of effort applied to tool-making and tool use. That manufacturing was competitive, over the period of the data, means that limiting values of h_d and h_p were achieved. Thus the mean relationship is to be found as the equation presenting consistent values for h_p, a and m. The best estimate would be achieved by: determining all forty-one sets of transformed data; finding, for each, the values of h_p, at the intercept and that of the limiting value evaluated from the estimates of a and m; fitting the two sets of estimates as functions of time; determining the intersection of the two which is the point with the same values for the intercept and the limit.

This procedure is demonstrated in principle using the four base years' data plotted in Figure (6) by reference to Table 1 which shows the values of the coefficients of equation (14) for the four years and the limiting values of h_p determined from a and m.

Table 1 Coefficients of equation (14) for the Figure 6 mappings

base year	intercept h_p	a	$\frac{a_{\text{year}}}{a_{1909}}$	m	limiting value $h_p = \frac{1}{2} + \frac{m}{2a}$
1909	0.36	0.25	1.00	0.08	0.66
1929	0.47	0.35	1.40	0.09	0.63
1939	0.57	0.49	1.96	0.085	0.59
1949	0.67	0.87	3.48	0.12	0.57

Inspection of Table 1 suggests that the transient equation providing an appropriate description of Solow's production data is

$$q = 0.58\left[1 + \frac{0.49}{0.09}(1 - 0.58)(1 - e^{-0.09t})\right] \quad (31)$$

4.6. First derivative of the productivity definition

Differentiating the productivity definition of equation (4), $\dot{q} = h_p p$, with respect to time gives

$$\ddot{q} = h_p \dot{p} \quad (32)$$

As a relationship derived solely from first principles, it has universal validity. Output rates increase as tools are brought into use (capital deepening) and productivity is increased. The relationship describes the changes occurring along the lines of constant h_d in Figures 2 and 3.

For competitive industries at any given level of technical progress, the limiting value of h_p will be approached; the $t \to \infty$ limit of equation (22), tends to a constant $\ddot{q} : \dot{p}$ ratio. For industries with advanced technical capability, the ratio will move closer towards its ultimate limiting value of one half. Different industries with their own levels of technical progress and maintenance, will, for this ratio, exhibit distinctive values.

Empirical measurements will demonstrate this simple relationship with extremely high levels of statistical significance. Non-competitive industries and those with little use for tools will not exhibit such a constancy of ratio.

In the introduction to the conference proceedings, *Productivity Growth and Economic Performance: Essays on Verdoorn's Law*, the editors state

> "Verdoorn's law, in its simplest form, refers to a statistical relationship between the long-run rate of growth of labour productivity and the rate of growth of output, usually for the industrial sector. The term 'Verdoorn coefficient' denotes the regression coefficient between the two variables" (McCombie, Pugno and Soro, 2003, p. 1).

Taking this assertion as a *de facto* definition, then the *Verdoorn coefficient* to which they refer must be either h_p or h_p^{-1}. Since Verdoorn (2002) first noted the empirical relationship, many efforts have been made to explain the underlying mechanisms. Since attempts to express transient physical behaviour in terms of equilibria, are doomed to inevitable failure, specific difficulties arise in comparing published empirical data with the present predictions. The more significant of these are examined in appendix C, "The Verdoorn relationship", where conventional hypotheses and empirical determinations appear to conflate the two interpretations of the Verdoorn coefficient.

Angeriz, McCombie and Roberts (2008, p.64) note that "the dynamic law can be derived directly from the static law by differentiating with respect to time", thereby identifying the static law as the productivity definition and the dynamic law as its first derivative. Traditional reporting is frequently in terms of *returns to scale*.[13] Measurements detecting the productivity defining equation (4) are interpreted as *constant returns to scale* and those detecting equation (32) as *increasing returns to scale*. Paradox is perceived when analysis of the data detects both relationships and the expectation is of there being only one

.

For 118 firms between 1983 and 2002, Hartigh, Langerak and Zegveld (2009) examine the "Verdoorn Law" relationship for a number of industries. They find that, for the majority of the firms and industries examined, the relationships have statistical significance levels better than 0.001. Wide variations in the values of the coefficient are observed and are consistent with the predictions of the present analysis.

4.7. The Verdoorn coefficient and the intercept of the production function

The proportion of effort directly committed to producing the final output is h_p. It occurs in all quantitative descriptions of the production process and their derivatives. Its ubiquity provides a critical test by which transient analysis might be falsified. Values present empirically should, for competitive industries, sharing the same technologies, be close to the limiting values. For industries in which productivity increase is not from the use of tools and for non-competitive industries, the values may differ, without the competitive pressures driving them to convergence.

The value derived from the Solow data is $h_p = 0.58$ (Table 1 and equation (31)). Table 2 presents published values for the $\ddot{q}:\dot{p}$ ratio. All of these lie close to the 0.58 value. In his Table 1, Verdoorn (2002) presents, for the USA, three different values for the $\ddot{q}:\dot{p}$ ratio: from 1869 to 1899 as 0.42; from 1899 to 1939 as 0.57; from 1924 to 1939 as 1.67. The value for the USA between 1924 and 1939 lies within the period of the Solow data. The values (Verdoorn, 2002, p.29. Table 2.1) of 0.6 for the percentage increases in annual production and 1.0 for the percentage increases in productivity represent $h_p = 0.6$.

For competitive industries, while data scatter and the many equation forms used for estimates of the $\ddot{q}:\dot{p}$ ratio demonstrate a range of values, central values lie close to that obtained, from the Solow data. For non-competitive industries the reported values (Hartigh, Langerak and Zegveld, 2009) for the ratio are not driven towards limiting values of h_p and do not match them. That the $\ddot{q}:\dot{p}$ ratio and the proportion of effort dedicated to production, h_p, are the same variable, is not falsified by the empirical evidence.

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[13] After proving that general statements of the production function imply constant returns to scale and perceiving it as an *imbroglio*, Georgescu-Roegen (1970, p.9) concludes "once we have untangled the imbroglio hatched by blind symbolism. The economics of production, its elementary nature notwithstanding, is not a domain where one runs no risk of committing some respectable errors. In fact, the history of every science, including that of economics, teaches us that the elementary is the hotbed of the errors that count most".
Kaldor (1972) discusses the scale effects of equipment dimension on performance and the difficulties in establishing appropriate mathematical representation of physical systems. While scale effects are important in equipment design, they are subsumed into mathematical representation as technical progress. This is not how scale is understood in current economic analysis. Kaldor (1972, p.1255) ends "The problem then becomes not just one of 'solving the mathematical difficulties' resulting from discontinuities but the much broader one of replacing the 'equilibrium approach' with some, as yet unexplored, alternative that makes use of a different conceptual framework".
Transient analysis, demonstrating returns to scale are constant, resolves the imbroglio and provides such a "different conceptual framework".

Table 2 Published values of the ratio $\ddot{q}:\dot{p}$

Author(s)	Area	Period	$\ddot{q}:\dot{p}$
Verdoorn (2002, p.29)	USA‡	1924 to 1939‡	0.6‡
Fingleton and McCombie (1998, p.92)	Europe	1979 to 1989	0.575
Vergeer and Kleinknecht (2007, p.35)	OECD	1960 to 2004	0.55
Hartigh, Langerak and Zegveld (2009, p.363)	Holland	1983 to 2002	0.672
Angeriz, McCombie and Roberts (2008, p.78)	Europe	1986 to 2002	0.579
Pons-Novell and Viladecans-Marsal (1999, p.448)	Europe	1984 to 1992	0.560, 0.587, 0.628

‡ Within the period of the Solow data

To summarise, the predictions of transient analysis are consistent with published empirical data. Critical tests of limiting values are confirmed. They are consistent with the underlying identity of the coefficients.

5. Conclusion

Transient mathematical relationships, derived from *first principles*, provide a *parsimonious quantitative description of the development histories of manufacturing projects and industries*.

Appendices

A. Labour-time

The use of an alternate representation for human effort is introduced in sub-section 2.1, "The defining relationships", item 2b. By rewriting equations (2) and (3) in terms of a single labour-time variable, H, the two equations and equation (35) become respectively

$$\frac{\mathrm{d}q}{\mathrm{d}H_p} = p, \qquad \frac{\mathrm{d}p}{\mathrm{d}H_d} = a, \qquad \frac{\partial^2 q}{\partial H_d \partial H_p} = a.$$

Integrating the central relationship and substituting the result into the leftmost form, produces a relationship analogous to equations (7) and (13),

$$p = 1 + aH_d$$

and therefore the total quantity of output is

$$q = H_p(1 + aH_d). \tag{33}$$

The total quantity of output produced over time period, Δt, is $\sum \Delta q_t$. The overall mean output rate at time t is therefore $(\sum \Delta q_t) \div \Delta t$. Empirical data is available in this form for extended periods of time, providing empirical estimates of the instantaneous output rate found by differentiating equation (33) with respect to time,

$$\dot{q} = \frac{\mathrm{d}}{\mathrm{d}t}[H_p(1 + aH_d)]. \tag{34}$$

Evaluation of equation (34) requires the use of the product rule. The derivative of H_p is h_p. Only if H_d can be expressed as a function of time, is it possible to find its derivative. Without knowing this relationship, the problem is intractable. The relationship *is known* from equation (12), *derived from an analysis separating* the h and t parameters, and thereby rendering parameterisation into H_d and H_p *irrelevant*.

An interesting corollary presents itself. By comparing equation (34) with $\dot{q} = f\,(a, l, k)$, the ubiquitous general statement of production functions in conventional analysis, leads to the singular conclusion that H_d is equivalent to k.

Conventional economic analysis implicitly asserts the equality $k \equiv H_d$ and thereby declares *capital to be labour-time*! The neoclassical production function and the labour theory of value are tautologies.

B. Dimensional analysis

The units of the present analysis are the *natural units* of the problem space and so subsume dimensional analysis. Equations (2) and (3) are axioms of the analysis. Substituting p from equation (2) into equation (3) gives

$$a = \frac{\partial^2 p}{\partial h_d \partial t} = \frac{\partial^2}{\partial h_d \partial t}\left(\frac{\partial^2 q}{\partial h_p \partial t}\right) = \frac{\partial^4 q}{\partial h_d \partial h_p \partial t^2}. \tag{35}$$

Dividing equation (35) by a establishes

$$\frac{\partial^4 q}{a \partial h_d \partial h_p \partial t^2} = 1 \tag{36}$$

with units $[QA^{-1}H^{-2}T^{-2}]=1$; 1 is the identity element of dimensionless groups.

The general equation $f(q, h_d, h_p, t, a) = 0$ may be used to specify a project's transient development history; as $h_d + h_p = 1$, h_d and h_p represent a single variable. With four independent variables Buckingham's Π theorem states that the relationship may be represented by a single dimensionless number Π; such that $F(\Pi) = 0$ where $\Pi = q^{i_q} h^{i_d} h^{i_p} t^{i_t} a^{i_a}$ and i_q, i_d, i_p, i_t, i_a are integer. By inspection of equation (36), dimensionless groups of the form $\left[\frac{q}{a h_d h_p t^2}\right] = c$ are appropriate, where c is an arbitrary constant whose value is determined from actual data. Buckingham's Π theorem does not provide a single dimensionless parameter, but a set from which suitable forms may be selected.

Equations (4), (14) and (23) are derived rigorously from first principles and so provide alternate appropriate representations of dimensionless groupings. Dividing each equation by their right hand sides gives three groups[14]

$$\left(\frac{\dot{q}}{h_p p}\right) = 1, \qquad \left[\frac{\dot{q}}{h_p(p_0 + a\eta(t))}\right] = 1, \qquad \frac{a}{(1 - mt_k)(p_0 + at_k)(t - t_k)} = 1. \tag{37}$$

The initial value of productivity, p_0, is presented to emphasise the unit rather than specific values. In the right hand group $(1 - mt_k) \equiv h_p$; $(h_d + h_p = 1,\ h_d \equiv mt_k)$.

The leftmost statement of equations (37) is the productivity definition of equation (4); the central is a statement of the instantaneous relationship of equation (14); the rightmost is the overall mean output relationship of equation (24).

C. The Verdoorn relationship

McCombie, Pugno and Soro (2003, p.5) observe "The story of the theoretical explanations of Verdoorn's Law is even more complicated, and far more open to debate". Fingleton (2001, p.7) describes a number of possible explanations of the relationship and states "Verdoorn's Law appears to be consistent with different theoretical positions or with different underlying technical relationships".

Without valid theoretical understanding, many alternative relationships, frequently without precise definition of the concepts being examined, hypothesise mechanisms to explain the empirical data. One common and the simplest statement (Kaldor (1975, p.891), McCombie and Ridder (1984, p.269), Fingleton and McCombie (1998, p.80), Angeriz, McCombie and Roberts (2008, p.65) etc.) is of the form

$$\dot{p} = \alpha + \beta\ddot{q} \tag{38}$$

where α is described as the exogenous rate of productivity increase and β as the Verdoorn coefficient.

Two possible interpretations of equation (38) and their implications are:

1. if the statement is a functional definition, then it is incomplete. Specifications of the domain and codomain are required.
2. if the statement is a differential equation, then integrating over the time interval [0, t], gives

$$p_t - p_0 = \alpha t + \beta(\dot{q}_t - \dot{q}_0).$$

Setting $\dot{q}_t = \dot{q}_0$ gives $p_t = p_0 + \alpha t$. If $\alpha \neq 0$ then, for no obvious technical reason, productivity increases indefinitely with time (implied by the description exogenous rate of productivity increase). Only if $\alpha = 0$ can equation (38) have any theoretical validity.

Both interpretations lead to *reductio ad absurdum* positions which invalidate any claim to theoretical validity for equation (38).

[14] A further dimensionless group may also be derived from equation (16).

Setting $\alpha = 0$ in equation (38) and comparing the result with equation (32) demonstrates that $\beta \equiv h_p^{-1}$, thereby revealing a further problem. By definition $0 < h_p \leq 1$, therefore $\beta \geq 1$ but some publications present theoretical relationships incorporating equation (38) but report values in the range $0 < \beta \leq 1$ (Kaldor (1975, p.891) requires only that $\beta > 0$). Thirlwall (1983, p.350) observes "*Productivity growth has exceeded output growth in every country*" (no emphasis in the original). This statement and values, for the Verdoorn coefficient being less than one, are *mutually exclusive*, which implies a mismatch between theoretical description and empirical observation. Suggesting that empirical values, reported as being less than one, are arrived at by the following. Productivity and output rates are determined separately (frequently using regression techniques to provide an averaging procedure). The ratio is then reported conventionally, presenting fractional values rather than values greater than one. Conformance to the mathematical relationships presented requires $\beta > 1$.

References

Allen, R. C. (2012). 'Technology and the great divergence: global economic development since 1820'. In: *Explorations in Economic History* 49(1), pp. 1–16. url: http://www.sciencedirect.com/science/article/pii/S0014498311000416

Angeriz, A., J. S. L. McCombie and M. Roberts (2008). 'New estimates of returns to scaleand spatial spillovers for EU Regional manufacturing, 1986–2002'. In: *International Regional Science Review* 31(1), pp. 62–87.

Badunenko, O. and V. Zelenyuk (2004). *Technological Change, Technological Catch-up, and Capital Deepening: Relative Contributions to Growth and Convergence During 90's; a Comment*. Univ. of Leuven, Department of Economics.

Barnett, W. (2004). 'Dimensions and economics: some problems'. In: *Quarterly Journal of Austrian Economics* 7(1), pp. 95–104.

Bénétrix, A. S., K. H. O'Rourke and J. G. Williamson (2015). 'The spread of manufacturing to the poor periphery 1870–2007'. In: *Open Economies Review* 26(1), pp. 1–37. url: http://dx.doi.org/10.1007/s11079-014-9324-x.

Bigo, V. and I. Negru (2014). 'Mathematical modelling in the wake of the crisis: a blessing or a curse? What does the economics profession say?' In: *Cambridge Journal of Economics* 38(2), pp. 329–347.

Blaug, M. (1998). 'Disturbing Currents in Modern Economics'. In: *Challenge* 41(3), pp. 11–34. eprint: http://dx.doi.org/10.1080/05775132.1998.11472030
url: http://dx.doi.org/10.1080/05775132.1998.11472030

Boland, L. A. (2005). 'Economics in time vs time in economics: building models so that time matters'. In: *History of Economic Ideas* 13(1), p. 121.

Bureau International des Poids et Mesures (2006: updated 2014). *The International System of Units (SI)*. 8th. International Committee for Weights and Measures.

Byrne, R. W. (2004). 'The manual skills and cognition that lie behind hominid tool use'. In: *The Evolution of Thought. Evolutionary Origins of Great Ape Intelligence*. Ed. by A. E. Russon and D. R. Begun. Cambridge University Press, pp. 31–44.

Cohen, A. J. and G. C. Harcourt (2003). 'Retrospectives: Whatever happened to the Cam bridge capital theory controversies?' In: *The Journal of Economic Perspectives* 17(1), pp. 199–214.

Crafts, N. (2012). 'British relative economic decline revisited: the role of competition'. In: *Explorations in Economic History* 49(1), pp. 17–29.
url: http://www.sciencedirect.com/science/article/pii/S0014498311000295

D'Adda, C. and A. E. Scorcu (2003). 'On the time stability of the output-capital ratio'. In: *Economic Modelling* 20(6), pp. 1175–1189.
url: http://www.sciencedirect.com/science/article/pii/S0264999302000810.

Felipe, J. and J. S. L. McCombie (2014). 'The Aggregate Production Function: "Not Even Wrong"'. In: *Review of Political Economy* 26(1), pp. 60–84.

Felipe, J. and F. M. Fisher (2003). 'Aggregation in production functions: what applied economists should know'. In: *Metroeconomica* 54(2-3), pp. 208–262.

Felipe, J. and J. S. L. McCombie (2013). *The Aggregate Production Function and the Measurement of Technical Change*. Edward Elgar Publishing.

Fingleton, B. and J. S. L. McCombie (1998). 'Increasing returns and economic growth: some evidence for manufacturing from the European Union regions'. In: *Oxford Economic Papers* 50(1), pp. 89–105.

Fingleton, B. (2001). 'Equilibrium and economic growth: spatial econometric models and simulations'. In: *Journal of Regional Science* 41(1), pp. 117–147.

Georgescu-Roegen, N. (1970). 'The economics of production'. In: *American Economic Review* 60(2), pp. 1–9.

Hahn, F. (1991). 'The Next Hundred Years'. In: The Economic Journal 101(404), pp. 47–50.
url: http://www.jstor.org/stable/2233836

Hardin, G. (1968). 'The tragedy of the commons'. In: *Science* 162(3859), pp. 1243–1248.

Hartigh, E. den, F. Langerak and M. Zegveld (2009). 'The Verdoorn law, firm strategy and, firm performance'. In: *Value-added Partnering and Innovation in a Changing World*. Ed. by M. van Geeuhaizen et al. Purdue University Press. Chap. 21, pp. 358–372.

Hogan, W. P. (1958). 'Technical progress and production functions'. In: *The Review of Economics and Statistics* 40, pp. 407–411.

Humphrey, T. M. (1997). 'Algebraic production functions and their uses before Cobb-Douglas'. In: *Economic Quarterly-Federal Reserve Bank of Richmond* 83, pp. 51–83.

Jensen, B. (2016). 'von Thünen: Capital, Production Functions, Marginal Productivity Wages, and the Natural Wage'. In: *German Economic Review*, pp. 1–30.

Jones, C. I. and P. M. Romer (2010). 'The New Kaldor Facts: Ideas, Institutions, Population, and Human Capital'. In: *American Economic Journal: Macroeconomics* 2(1).
url: http://www.ingentaconnect.com/content/aea/aejma/2010/00000002/00000001/art00009

Kaldor, N. (1961). 'Capital accumulation and economic growth'. In: *The Theory of Capital: Proceedings of a Conference held by the International Economic Association*. Ed. by D. C. Hague. London: Macmillan & Co Ltd. Chap. 10, pp. 177–222.

Kaldor, N. (1972). 'The irrelevance of equilibrium economics'. In: *The Economic Journal* 82(328), pp. 1237–1255. url: http://www.jstor.org/stable/2231304

Kaldor, N. (1975). 'Economic growth and the Verdoorn law–a comment on Mr Rowthorn's article'. In: *The Economic Journal* 85(340), pp. 891–896.

Koop, G., J. Osiewalski and M. F. J. Steel (1999). 'The Components of Output Growth: A Stochastic Frontier Analysis'. In: *Oxford Bulletin of Economics and Statistics* 61(4), pp. 455–487. url: http://dx.doi.org/10.1111/1468-0084.00139.

Koop, G., J. Osiewalski and M. F. J. Steel (2000). 'Modeling the Sources of Output Growth in a Panel of Countries'. In: *Journal of Business & Economic Statistics* 18(3), pp. 284–299.
eprint: http://www.tandfonline.com/doi/pdf/10.1080/07350015.2000.10524871
url: http://www.tandfonline.com/doi/abs/10.1080/07350015.2000.10524871

Kumar, S. and R. R. Russell (2002). 'Technological change, technological catch-up, and capital deepening: relative contributions to growth and convergence'. In: *American Economic Review*, pp. 527–548.

Lavoie, M. (2008). 'Neoclassical empirical evidence on employment and production laws as artefact'. In: *Economía Informa* 351, pp. 9–36.

Lazzarini, A. (2011). *Revisiting the Cambridge Capital Theory Controversies: A Historical and Analytical Study.* Pavia University Press.

Madsen, J. B., V. Mishra and R. Smyth (2012). 'Is the output-capital ratio constant in the very long run?' In: *The Manchester School* 80(2), pp. 210–236.

Makieła, K. (2014). 'Bayesian Stochastic Frontier Analysis of Economic Growth and Productivity Change in the EU, USA, Japan and Switzerland'. In: *Central European Journal of Economic Modelling and Econometrics* 6(3), pp. 193–216.

Mayumi, K. and M. Giampietro (2010). 'Dimensions and logarithmic function in economics: a short critical analysis'. In: *Ecological Economics* 69(8), pp. 1604–1609.

McCombie, J. S. L. and J. R. de Ridder (1984). '"The Verdoorn law controversy": some new empirical evidence using U.S. state data'. In: *Oxford Economic Papers* 36(2), pp. 268–284.
eprint: http://oep.oxfordjournals.org/content/36/2/268.full.pdf+html
url: http://oep.oxfordjournals.org/content/36/2/268.short

McCombie, J. S. L. (2000). 'The Solow residual, technical change, and aggregate production functions'. In: *Journal of Post Keynesian Economics*, pp. 267–297.

McCombie, J. S. L., M. Pugno and B. Soro (2003). *Productivity Growth and Economic Performance: Essays on Verdoorn's Law*. Palgrave Macmillan.

Mishra, S. K. (2010). 'A brief history of production functions'. In: *The IUP Journal of Managerial Economics* VIII(4), pp. 6–34.

North, D. C. (1994). 'Economic Performance Through Time'. In: *The American Economic Review* 84(3), pp. 359–368. url: http://www.jstor.org/stable/2118057

Pons-Novell, J. and E. Viladecans-Marsal (1999). 'Kaldor's laws and spatial dependence: evidence for the European regions'. In: *Regional Studies* 33(5), pp. 443–451.

Robinson, J. (1953). 'The production function and the theory of capital'. In: *The Review of Economic Studies* 21(2), pp. 81–106.

Robinson, J. (1971). 'Capital theory up to date: a reply'. In: *The Canadian Journal of Economics/Revue Canadienne d'Economique* 4(2), pp. 254–256.

Robinson, J. (1980). 'Time in economic theory'. In: *Kyklos* 33(2), pp. 219–229.

Rodrik, D. (2013). 'Unconditional convergence in manufacturing'. In: The *Quarterly Journal of Economics* 128(1), pp. 165–204.
eprint: http://qje.oxfordjournals.org/content/early/2013/01/21/qje.qjs047.full.pdf+html.
url: http://qje.oxfordjournals.org/content/early/2013/01/21/qje.qjs047.abstract

Salas, S. L., E. Hille and G. J. Etgen (2007). *Calculus: One and Several Variables*. 10th. Wiley New York.

Setterfield, M. (1995). 'Historical time and economic theory'. In: *Review of Political Economy* 7(1), pp. 1–27.

Setterfield, M. and S. G. Suresh (2014). 'Aggregate structural macroeconomic analysis: a reconsideration and defence'. In: *Cambridge Journal of Economics*, pp. 797–815. Smith, A. (1776). *An Inquiry into the Nature and Causes of the Wealth of Nations*. Project

Gutenberg (2013). url: https://www.gutenberg.org/ebooks/3300

Solow, R. M. (1955). 'The production function and the theory of capital'. In: *The Review of Economic Studies* 23(2), pp. 101–108.

Solow, R. M. (1957). 'Technical change and the aggregate production function'. In: *The Review of Economics and Statistics* 39(3), pp. 312–320.

Solow, R. M. (1958). 'Technical progress and production functions: reply'. In: *The Review of Economics and Statistics* 40(4), pp. 411–413. url: http://www.jstor.org/stable/1926346.

Solow, R. M. (1994). 'Perspectives on growth theory'. In: *The Journal of Economic Perspectives* 8(1), pp. 45–54. url: http://www.jstor.org/stable/2138150.

Thirlwall, A. (1983). 'A plain man's guide to Kaldor's growth laws'. In: *Journal of Post Keynesian Economics* 5(3), pp. 345–358. url: http://www.jstor.org/stable/4537750

Verdoorn, P. J. (2002). 'Factors that determine the growth of labour productivity'. In: *Productivity Growth and Economic Performance*. Ed. by J. S. McCombie, M. Pugno and B. Soro. Basingstoke: Palgrave Macmillan, pp. 28–36.

Vergeer, R. and A. Kleinknecht (2007). 'Jobs versus productivity? The causal link from wages to labour productivity growth'. In: *TU Delft Innovation Systems Discussion Papers*, IS 1.

Zamagni, S. and E. Agliardi (2004). *Time in Economic Theory*. 3 vols. Cheltenham: Edward Elgar Publishing Limited.

Zellner, A. and H. Ryu (1998). 'Alternative functional forms for production, cost and returns to scale functions'. In: *Journal of Applied Econometrics* 13(2), pp. 101–127.

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